

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

$$\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial uv}{\partial y}+\frac{\partial uw}{\partial z}=fv-\frac{1}{\rho_0}\frac{\partial p}{\partial x}+$$

$$\frac{\partial}{\partial x}(A_H\frac{\partial u}{\partial x})+\frac{\partial}{\partial y}(A_H\frac{\partial u}{\partial y})+\frac{\partial}{\partial z}(A_V\frac{\partial u}{\partial z})$$

$$\frac{\partial v}{\partial t}+\frac{\partial uv}{\partial x}+\frac{\partial v^2}{\partial y}+\frac{\partial vw}{\partial z}=-fu-\frac{1}{\rho_0}\frac{\partial p}{\partial y}+$$

$$\frac{\partial}{\partial x}(A_H\frac{\partial v}{\partial x})+\frac{\partial}{\partial y}(A_H\frac{\partial v}{\partial y})+\frac{\partial}{\partial z}(A_V\frac{\partial v}{\partial z})$$

$$\frac{\partial p}{\partial z}=-\rho g$$

$$\frac{\partial T}{\partial t}+\frac{\partial uT}{\partial x}+\frac{\partial vT}{\partial y}+\frac{\partial wT}{\partial z}=\frac{\partial}{\partial x}(K_H\frac{\partial T}{\partial x})+\frac{\partial}{\partial y}(K_H\frac{\partial T}{\partial y})+$$

$$\frac{\partial}{\partial z}(K_V\frac{\partial T}{\partial z})$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\begin{aligned}\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z} &= fv - \frac{1}{\rho_0} \frac{\partial p}{\partial x} + \\ \frac{\partial}{\partial x} (A_H \frac{\partial u}{\partial x}) &+ \frac{\partial}{\partial y} (A_H \frac{\partial u}{\partial y}) + \frac{\partial}{\partial z} (A_V \frac{\partial u}{\partial z})\end{aligned}$$

$$\begin{aligned}\frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y} + \frac{\partial vw}{\partial z} &= -fu - \frac{1}{\rho_0} \frac{\partial p}{\partial y} + \\ \frac{\partial}{\partial x} (A_H \frac{\partial v}{\partial x}) &+ \frac{\partial}{\partial y} (A_H \frac{\partial v}{\partial y}) + \frac{\partial}{\partial z} (A_V \frac{\partial v}{\partial z})\end{aligned}$$

$$\frac{\partial p}{\partial z} = -\rho g$$

$$\begin{aligned}\frac{\partial T}{\partial t} + \frac{\partial uT}{\partial x} + \frac{\partial vT}{\partial y} + \frac{\partial wT}{\partial z} &= \frac{\partial}{\partial x} (K_H \frac{\partial T}{\partial x}) \\ &+ \frac{\partial}{\partial y} (K_H \frac{\partial T}{\partial y}) + \frac{\partial}{\partial z} (K_V \frac{\partial T}{\partial z})\end{aligned}$$

$$\begin{aligned}\frac{\partial S}{\partial t} + \frac{\partial uS}{\partial x} + \frac{\partial vS}{\partial y} + \frac{\partial wS}{\partial z} &= \frac{\partial}{\partial x} (K_H \frac{\partial S}{\partial x}) \\ &+ \frac{\partial}{\partial y} (K_H \frac{\partial S}{\partial y}) + \frac{\partial}{\partial z} (K_V \frac{\partial S}{\partial z})\end{aligned}$$

$$\rho = \rho(T, S) = \frac{P}{\alpha + 0.698P}$$

where

$$P = 5890 + 38T - 0.375T^2 + 3S,$$

$$\alpha = 1779.5 + 11.25T - 0.0745T^2 - (3.8 + 0.01T)S$$

Steady 2D Flows: Stream function $\psi(x, y)$:

$$\frac{dx}{dt} = \frac{\partial \psi}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial \psi}{\partial x}$$

Fact: **Instantaneous stream functions (contours of ψ) are particle trajectories.**

Strategy:

Determine stability of stagnation (equilibrium) points

Find directions of compression and stretching (stable and unstable manifolds)

Strategy works for very special time-dependent systems (periodic perturbations), but fails for general time-dependent vector fields.

$$\psi(x,y,t)=\frac{A}{k}\sin \pi y \sin kx + k\epsilon \cos \omega t \cos kx$$

Duffing's Equation

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = x - x^3 + \epsilon \sin t$$

The flow is incompressible ($\operatorname{div} \mathbf{v} = 0$) so there is a streamfunction $\psi(x, y, t)$ such that

$$v_1 = -\frac{\partial \psi}{\partial y}, \quad v_2 = \frac{\partial \psi}{\partial x}$$

$$\psi = -\frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{1}{4}x^4 + \epsilon x \sin t$$